Chapter 6: Point Estimation

6.4 Maximum Likelihood Estimation

Recall:

- The sample mean ______ can be thought of as an estimate of the distribution mean ______
- The sample variance ______ can be used as an estimate of the distribution variance ______

In this section we are going to address the following questions:

1. How good are these estimates?
2. What makes an estimate good?
3. Can we say anything about the closeness of an estimate to an unknown parameter?

Here we consider random variables for which the functional form of the *pmf* or *pdf* is known, but the distribution depends on an unknown parameter (say, θ) that may have any value in a set (say, Ω) called the ______________________.

Example 1.

Note 1. – **Estimator** of θ: Statistic $u(X_1, X_2, \ldots, X_n)$ where $X_1, X_2, \ldots, X_n$ are random variables.

– **Estimate** of θ: $u(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are observations.

Example 2. 1. Suppose that X is $b(1, p)$, so that the *pmf* of X is:

2. We note that ________________, where ______ represents the parameter space. That is, the space of all possible values of the parameter p.

3. Given a random sample $X_1, X_2, \ldots, X_n$, the problem is to find an estimator $u(X_1, X_2, \ldots, X_n)$ such that $u(x_1, x_2, \ldots, x_n)$ is a good point estimate of p, where $x_1, x_2, \ldots, x_n$ are the observed values of the random sample.

4. Now, the probability that $X_1, X_2, \ldots, X_n$ takes these particular values is:

5. One reasonable way to proceed toward finding a good estimate of p is to regard this probability (or joint *pmf*) as a function of p and find the value of p that **maximizes** it.

6. The joint **pmf**, when regarded as a function of p, is frequently called the ________________ function.

7. Thus, here the likelihood function is:

8. To find the estimator that maximizes the likelihood function, first take *ln* of the likelihood function and then take the derivative and set it equal to zero to find the critical values.

Example 3. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution. Find the maximum likelihood estimator of θ

Example 4. Suppose $X_1, X_2, \dots, X_n$ is a random sample from the geometric distribution. Find the maximum likelihood estimator of p

Example 5. Suppose $X_1, X_2, \dots, X_n$ is a random sample from the $N(\mu, \sigma^2)$. Find the maximum likelihood estimators of μ and σ.

Definition 1. *If $E[u(X1, X2, ..., Xn)] = \theta$, then the statistic $u(X1, X2, ..., Xn)$ is called an* ***unbiased*** *estimator of θ. Otherwise, it is said to be* ***biased****.*

Example 6. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the $N(\mu, \sigma^2)$.

1. Show that $\bar{X}$ is an unbiased estimator of μ
2. Show that s^2 is an unbiased estimator of σ^2

Method of Moments Estimators: The method of moments involves equating sample moments with theoretical moments. So, let's start by making sure we recall the definitions of theoretical moments, as well as learn the definitions of sample moments.

1. Equate the first sample moment about the origin $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ to the first theoretical moment $E(X)$.
2. Equate the second sample moment about the origin $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ to the first theoretical moment $E(X^2)$.
3. Continue equating sample moments about the origin, M_k, with the corresponding theoretical moments $E(X^k)$, $k = 3, 4, ...$ until you have as many equations as you have parameters.
4. Solve for the parameters.

Example 7. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n with the pdf $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$. Use the method of moments to find a point estimate for θ

Example 8. Let the distribution of X be $N(\mu, \sigma^2)$. Use the method of moments to find a point estimates for μ and σ^2