Chapter 6: Point Estimation

6.3 Order Statistics

Order statistics are the observations of the random sample, arranged, or ordered, in magnitude from the smallest to the largest.

Assumptions:

1. n independent observations come from a continuous-type distribution.
2. the probability of any two observations being equal is zero.

Example 1. The values $x_1 = 0.62, x_2 = 0.98, x_3 = 0.31, x_4 = 0.81, and x_5 = 0.53$ are the $n = 5$ observed values of five independent trials of an experiment with pdf $f(x) = 2x$, $0 < x < 1$.

(a) Find the order statistics.

(b) Find the median and range of the sample.

How to determine the **cdf** of the r^{th} order statistic, Y_r?

If $X_1, X_2, ..., X_n$ are observations of a random sample of size n from a continuous-type distribution, we let the random variables $Y_1 < Y_2 < ... < Y_n$ denote the order statistics of that sample. That is,

Y_1 = smallest of $X_1, X_2, ..., X_n$,

Y_2 = second smallest of $X_1, X_2, ..., X_n$,

.

.

Y_n = largest of $X_1, X_2, ..., X_n$.

Let $Y_1 < Y_2 < ... < Y_n$ be the order statistics of n independent observations from a distribution of the continuous type with cdf $F(x)$ and pdf $F'(x) = f(x)$, where $0 < F(x) < 1$ for $a < x < b$ and $F(a) = 0$, $F(b) = 1$.

(It is possible that $a = \infty$ and/or $b = +\infty$.) The event that the r^{th} order statistic Y_r is at most y, $\{Y_r \leq y\}$, can occur if and only if at least r of the n observations are less than or equal to y. That is, here the probability of "success" on each trial is $F(y)$, and we must have at least r successes.

The **CDF** of Y_r

The **PDF** of Y_r

Example 2. – Find the pdf of the smallest order statistic

– Find the pdf of the largest order statistic

Example 3: Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of five independent observations from an exponential distribution that has a mean of $\theta = 3$.

– Find the pdf of the sample median.

– Compute the probability that Y_4 is less than 5.

– Determine $P(1 < Y_1)$.

Example 4: Let $Y_1 < Y_2 < \ldots < Y_{19}$ be the order statistics of $n = 19$ independent observations from the exponential distribution with mean θ.

- What is the pdf of Y_1?
- Find the value of $E[F(Y_1)]$, where F is the cdf of the exponential distribution.

Example 5: Let $W_1 < W_2 < \ldots < W_n$ be the order statistics of n independent observations from a $U(0, 1)$ distribution.

- Find the pdf of W_1 and that of W_n.
- Use the results of (a) to verify that $E(W_1) = 1/(n + 1)$ and $E(W_n) = n/(n + 1)$.